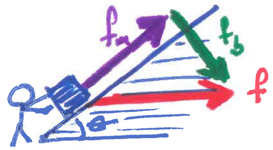
Truss Systems & Matrices (Intro)

Idea: Generalize basic physics problem:

Box pushed with force $\underline{f}$
on slope of angle θ
Split force into components

$\underline{f}_m$ = force parallel to slope (causing motion)

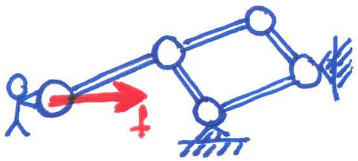
$\underline{f}_b$ = force perpendicular to slope (balanced by system)



$$\underline{f} = \underline{f}_m + \underline{f}_b$$

$$|\underline{f}_m| = |\underline{f}| \cos\theta$$
$$|\underline{f}_b| = |\underline{f}| \sin\theta$$

Now instead of pushing against a slope, force is applied to a mobile system of "trusses" and "joints" & force distributes throughout system causing motion at joints & tension/compression on trusses



$$\underline{f} = \underline{f}_m + \underline{f}_b$$

f_b = force "parallel" to system
— balanced by tension/compression of bars

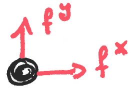
f_m = force "perpendicular" to system
— unbalanced by bars — causing motion

Truss Systems

A truss system consists of

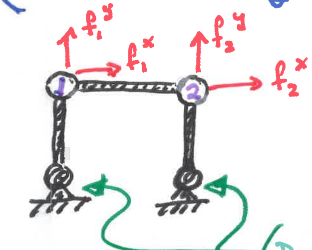
(All bars have a node at each end.
Nodes cannot be attached to the middle of a bar.)

Forces are applied (or measured) at each node, with both a horizontal & vertical component.



$$\underline{f} = \begin{bmatrix} f^x \\ f^y \end{bmatrix}$$

(We ignore forces on fixed nodes, since they are immediately balanced from outside the system.)

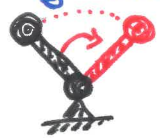


$$\underline{f} = \begin{bmatrix} f_1^x \\ f_1^y \\ f_2^x \\ f_2^y \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{Node \#1 force} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{Node \#2 force} \end{matrix}$$

(Do not apply/measure force on fixed nodes.)

We ignore torsion forces at nodes. This is equivalent to assuming bars can freely rotate.

Bars can rotate around nodes



Nodes that are not fixed can move

Questions:

① Force splitting. Given a force vector $\underline{\underline{f}}$, how can you split it into components

$$\underline{\underline{f}} = \underline{\underline{f}}_b + \underline{\underline{f}}_m \quad ?$$

balanced — by tension/comp. force in bars

unbalanced — causing motion of system (or torsion at nodes)

→ Which forces are balanced / unbalanced?

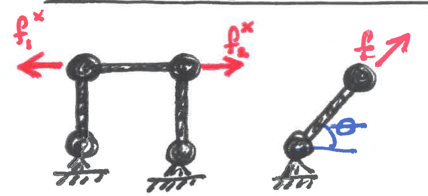
→ In which directions can forces move system?

② Stability. If all possible forces are balanced ($\underline{\underline{f}} = \underline{\underline{f}}_b$) then we say a truss system is "stable". How do we check this?

③ Efficiency. If a bar can be removed without changing #balanced forces or #directions system can move, then system is not efficient (it has unneeded bars).

[A single matrix can be used to answer all of these questions! These questions all relate to different properties of the matrix.]

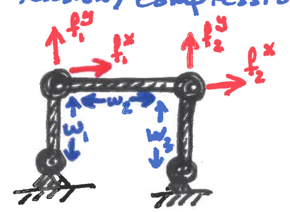
The Force Balance Matrix, $B = A^T$



$$\underline{\underline{f}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{\underline{f}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

A force is balanced if it purely pushes/pulls in direction of trusses — causing tension/compression of bars.

Thinking in reverse, a force is balanced if it can be written as a sum of internal tension/compression forces in bars.



w_1 = tension in bar #1

w_2 = tension in bar #2

w_3 = tension in bar #3

$$\left\{\begin{aligned} f_1^x &= -w_2 \\ f_1^y &= w_1 \\ f_2^x &= w_2 \\ f_2^y &= w_3 \end{aligned}\right\} \Rightarrow \boxed{\begin{bmatrix} f_1^x \\ f_1^y \\ f_2^x \\ f_2^y \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}$$

(Note: Tension in bar 2 pushes node 1 in negative x-direction.)

$$\underline{\underline{f}}_b = B \, \underline{\underline{w}}$$

balanced force at nodes

"Force-Balance Matrix"

internal tension force in bars

Tension → Force (Multiply)

The vector $\underline{\underline{w}}$ of tensions of bars is equivalent to a vector $\underline{f}_b = B\underline{\underline{w}}$ of forces at nodes.

The tension vector $\underline{\underline{w}}$ balances the force vector $-B\underline{\underline{w}}$

Force → Tension (Divide)

A vector $\underline{f}$ of forces at nodes is balanced precisely when $B\underline{\underline{w}} = \underline{f}$ can be solved for $\underline{\underline{w}}$.

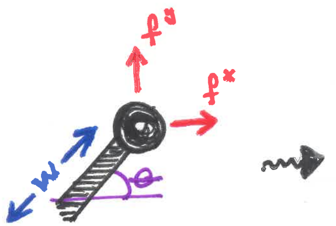
(Usually $B\underline{\underline{w}} = \underline{f}$ will not have a solution.)

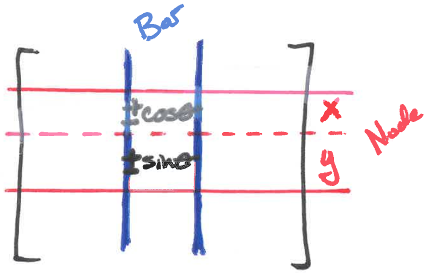
The force-balance matrix B is very simple:

$$B\,\underline{\underline{w}} = \underline{f}_b$$

B has input (col) for each bar

B has two outputs (rows) for each node (x & y dir.)





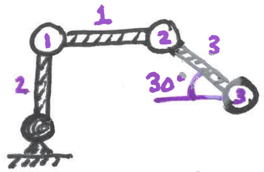
Entries of B record meetings of bars & nodes

they are 0 if bar does not touch node

otherwise $\begin{bmatrix} \pm\cos\theta \\ \pm\sin\theta \end{bmatrix}$ where θ is angle of bar.

(The ± is determined by whether moving from the bar to node is + or − direction)

EX Write the force-balance matrix for



(using the given order for bars & nodes)

$$B = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \text{Bar 1} & \text{Bar 2} & \text{Bar 3} \end{array}$$

$$B = \left[\begin{array}{c:c:c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 1 & 0 & -\cos 30 \\ 0 & 0 & \sin 30 \\ \hdashline 0 & 0 & \cos 30 \\ 0 & 0 & -\sin 30 \end{array}\right] \begin{array}{cl} x & \\ y & \text{Node 1} \\ x & \\ y & \text{Node 2} \\ x & \\ y & \text{Node 3} \end{array} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -\sqrt{3}/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & \sqrt{3}/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

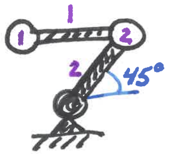
Bar #1 connects node #1 & #2 in x-direction

Bar #2 connects node #1 & outside in y-direction

Bar #3 connects node #2 & #3 in 30° angle

(Note: Each bar can connect at most two moving nodes — so most entries in the columns of B are usually 0)

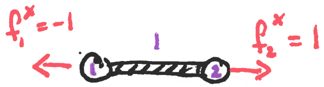
EX Write the force balance matrix for



(using the given order for bars & nodes)

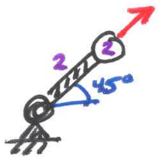
$$B = \begin{array}{c} \begin{matrix}\text{Bar 1} & \text{Bar 2}\end{matrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & \cos 45 \\ 0 & \sin 45 \end{bmatrix} \end{array} \begin{array}{l} \left.\begin{matrix} x \\ y \end{matrix}\right] \text{Node 1} \\ \left.\begin{matrix} x \\ y \end{matrix}\right] \text{Node 2} \end{array} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix}$$

Note: Each column of B is a force that the given bar balances.



bar 1 balances force $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

column 1 of B



bar 2 balances force $\begin{bmatrix} 0 \\ 0 \\ \cos 45 \\ \sin 45 \end{bmatrix}$

column 2 of B

If B has columns $\underline{b}_1, \dots$ then balanced forces are

$$B = \begin{bmatrix} | & | & | & \\ b_1 & b_2 & b_3 & \cdots \\ | & | & | & \end{bmatrix} \Longrightarrow \underline{f}_b = B\underline{w} = \begin{bmatrix} | & | & \\ b_1 & b_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \end{bmatrix}$$

$$= \underline{b}_1 w_1 + \underline{b}_2 w_2 + \cdots$$

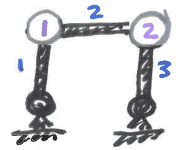
$$\{\text{Balanced forces, } \underline{f}_b\} = \{B\underline{w}\}$$

combinations of columns of B

$$= \{\underline{b}_1 w_1 + \underline{b}_2 w_2 + \cdots\}$$

$$= \text{"Column Space of } B\text{"}$$

EX: Which forces are balanced?



A) $\underline{f} = \begin{bmatrix} 3 \\ -2 \\ -3 \\ 0 \end{bmatrix}$ B) $\underline{f} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

Force balance matrix is $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A) Try to solve $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \\ 0 \end{bmatrix}$
$\rightarrow w_2 = -3$
$\rightarrow w_1 = -2$
$\rightarrow w_2 = -3$
$\rightarrow w_3 = 0$

$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$ Balanced.

B) Try to solve $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$
$\rightarrow w_2 = 0$
$\rightarrow w_1 = 1$
$\rightarrow w_2 = 1$
$\rightarrow w_3 = -1$
!

No solution. Not balanced.
